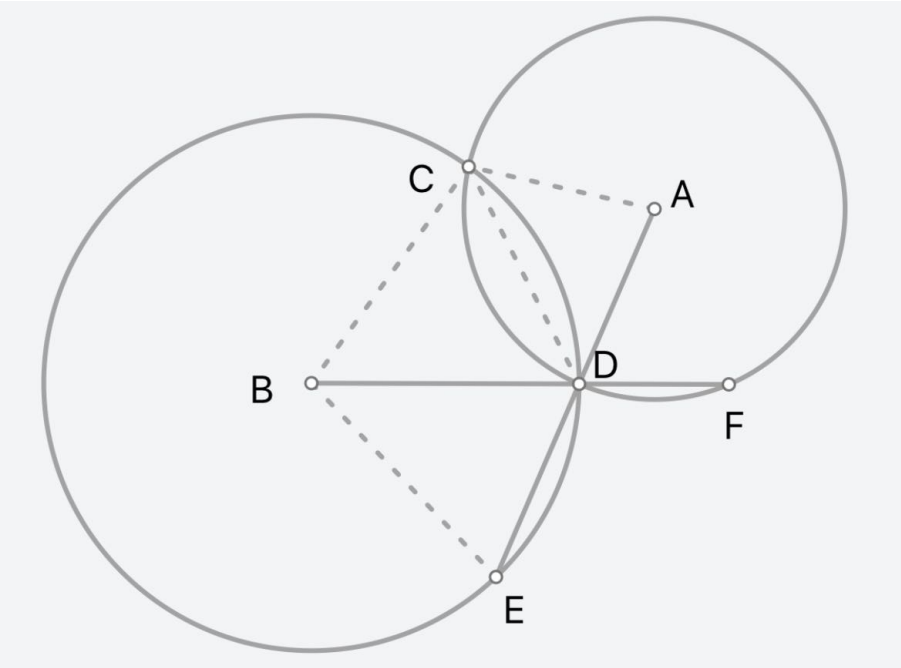
Euclidea 附录

A 附录

A.1 命题

4.2.1

> 设圆 A 与 圆 B 交于点 C 与 D, 延长 AD 交圆 B 于点 E, 延长 BD 交圆 A 于点 F, 则 A, C, B, E, F 五点共圆.



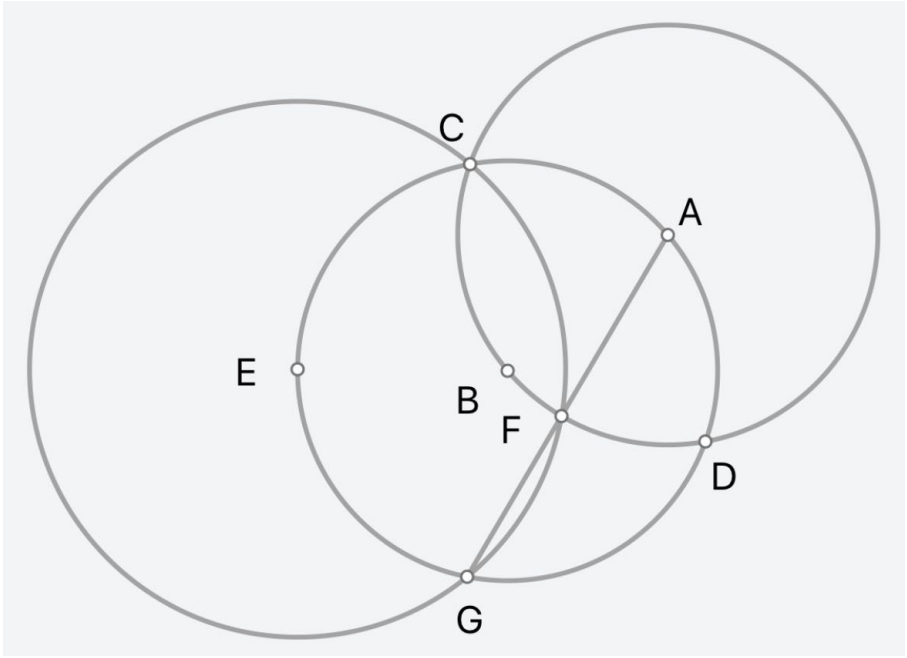
证明

$\angle A = 180^\circ - 2\angle CDA = 2\angle CDE - 180^\circ = 180^\circ - \angle B$, 故 A, C, B, E 四点共圆. 同理 A, C, B, E, F 五点共圆. 证毕.

注 当两圆相切时, 退化为四点共线 (广义圆).

4.2.2

> 圆 A 上有一点 B, 以 B 为圆心, BA 为半径作圆, 交圆 A 于点 C, D. 取圆 B 上另一点 E, 以 E 为圆心, EC 为半径作圆, 交圆 A 于点 F, 交圆 B 于点 G, 则 A, F, G 三点共线.



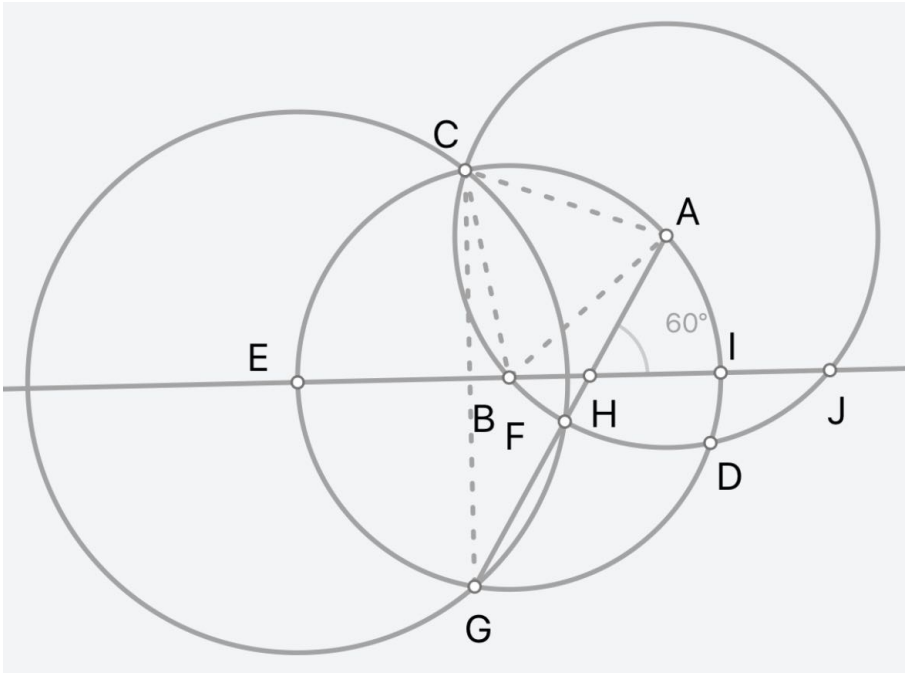
证明

设 AF 交圆 E 于点 G', 则由命题 4.2.1 知 $ACEG'$ 四点共圆. 而点 A, C, E 都在圆 B 上, 故点 G' 也在圆 B 上, 即 G' 是圆 B 与圆 E 的交点, 故 G' 与 G 重合, 即 AFG 三点共线. 证毕.

推论 同理 E, F, D 三点共线, 由此可得 $AF \cdot FG = EF \cdot FD$ 等.

4.2.3

> 同 4.2.2 条件, 作直线 BE, 交 AG 于点 H, 交圆 B 于点 I, 交圆 A 于点 J. 则 $\angle AHI = 60^\circ$.



证明

$\triangle ABC$ 是等边三角形, 故 $\angle CGA = \dfrac{1}{2}\angle CFA = 30^\circ$.

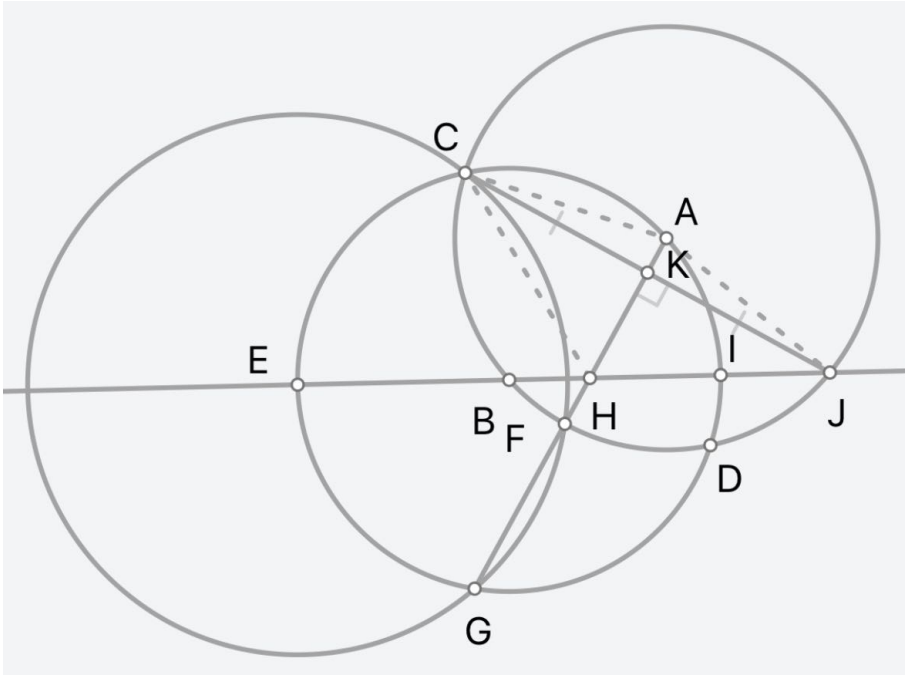
$CG \perp EB$, 故 $\angle AHI = \angle GHE = 90^\circ - \angle CFA = 60^\circ$.

证毕.

注 即 4E 作法.

4.2.4

> 同命题 4.2.3 条件, AH 是 CJ 的中垂线.



证明

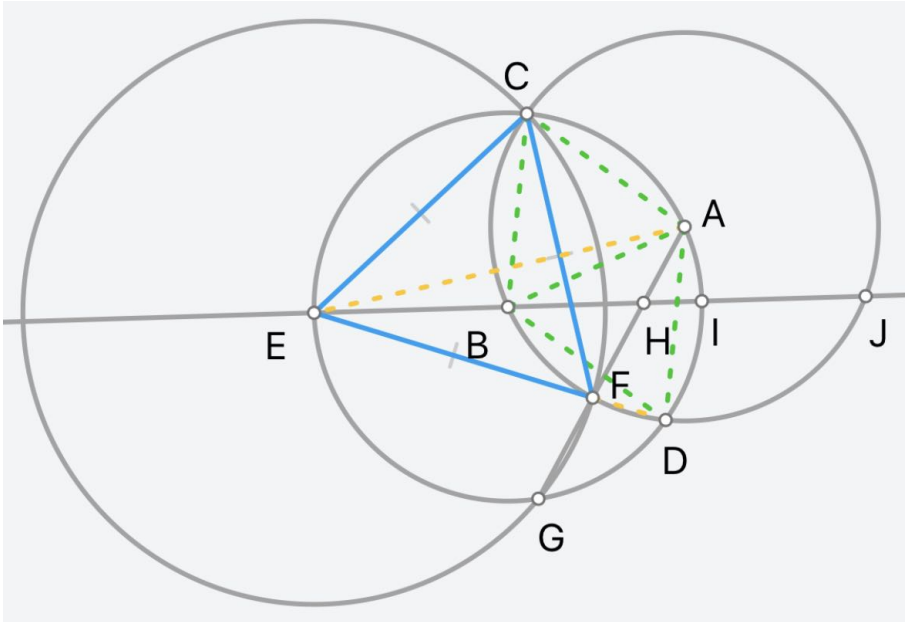
$\angle AHC = 2\angle AGC = 60^\circ = \angle AHJ$, 故 AH 是 CJ 是中垂线. (严谨的证明可以连接 CJ 交 AH 于点 K, 证明 $\triangle HCA \cong \triangle HJA$, $\triangle HCK \cong \triangle HJK$.)

证毕.

注 即 3L 作法.

4.2.5

> 同命题 4.2.3 条件, $\triangle ECF$ 是等边三角形.



证明

由 4.2.2 知: D, E, F 三点共线.

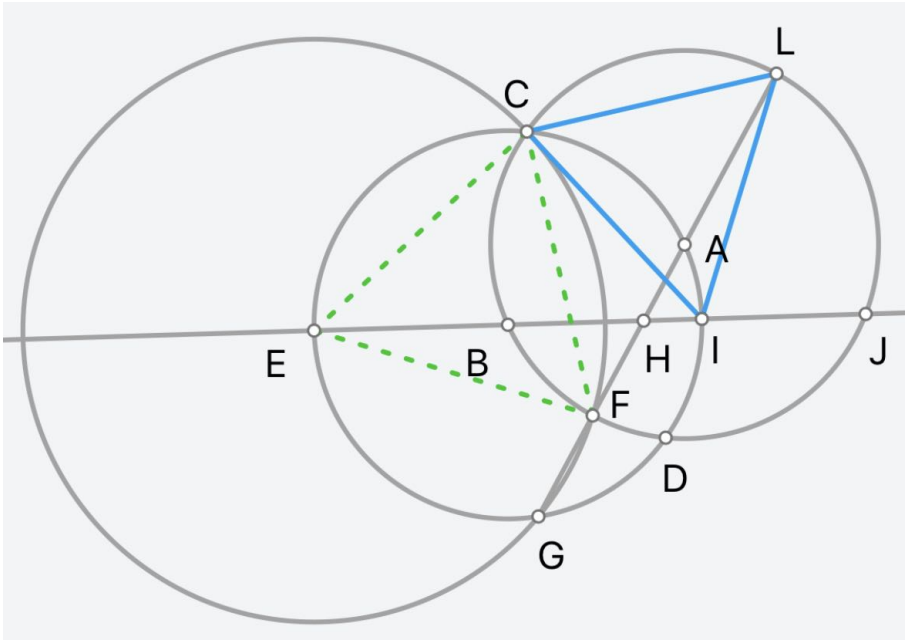
$$\angle CED = \angle AEC + \angle AED = \frac{1}{2}(\angle ABC + \angle ABD) = 60^\circ,\ EC = EF,$$

故 $\triangle ECF$ 是等边三角形. 证毕.

推论 点 C, E, J 都在以 F 为圆心的圆上.

4.2.6

> 同命题 4.2.3 条件, 延长 FA 交圆 A 于点 L, 则 $\triangle CIL$ 是等边三角形.



证明

由 4.2.5 知: $\triangle ECF$ 是等边三角形.

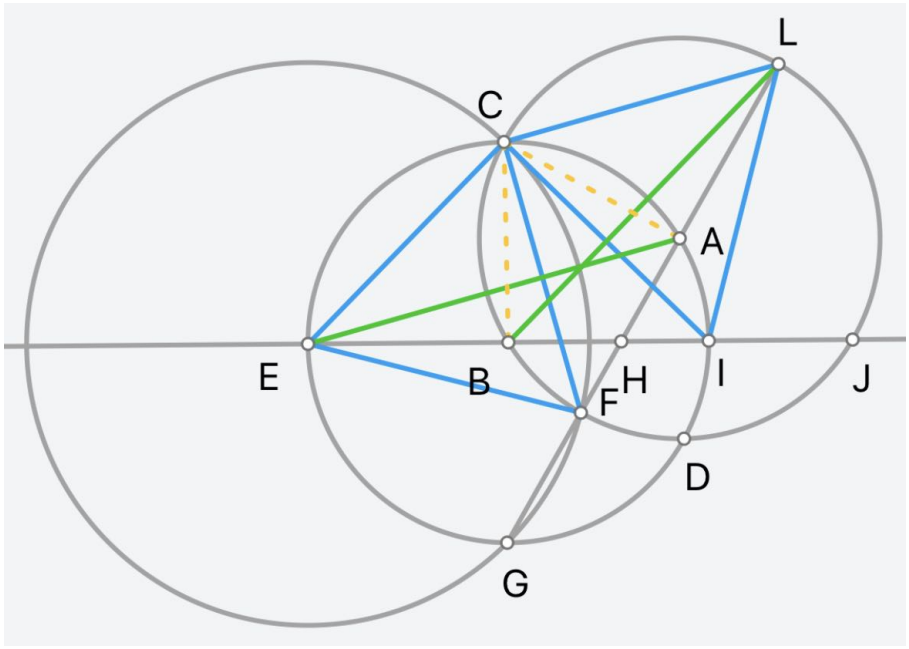
$$\angle ICL = \angle FCL - \angle CFI = \angle ECI - \angle FCI = \angle ECF = 60^\circ,\ IC = IL,$$

故 $\triangle CIL$ 是等边三角形. 证毕.

推论 点 C, L, G 都在以 I 为圆心的圆上.

4.2.7

> 同命题 4.2.6 条件, 则 AE 垂直平分 CF, BL 垂直平分 CI.



证明

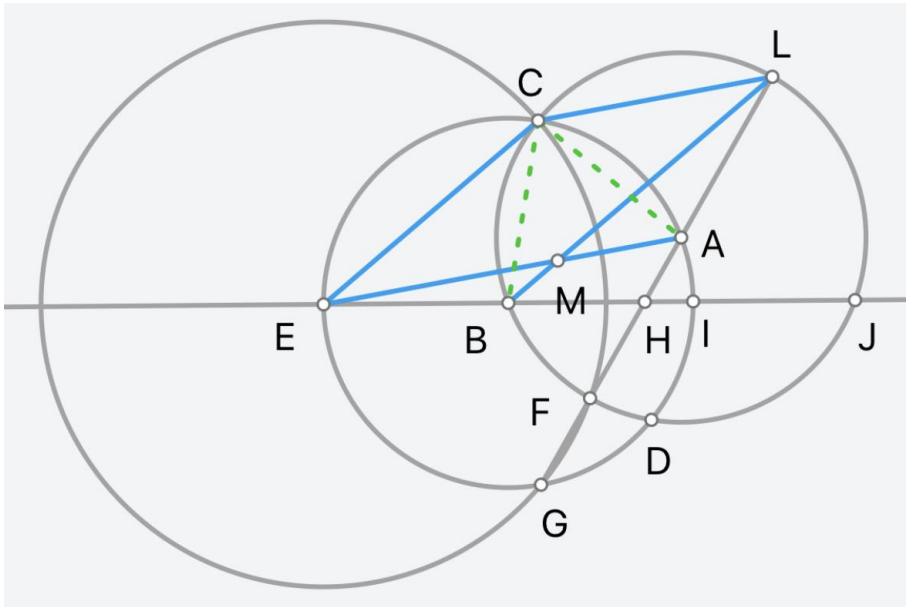
由 4.2.5 和 4.2.6 知: $\triangle ECF$ 和 $\triangle CIL$ 是等边三角形.

$BI = BC = AC = AF$, 故有题述垂直平分关系. 证毕.

注 至此可以得到许多全等或相似关系, 如 $\triangle ECB \cong \triangle FCA$, $\triangle ECI \cong \triangle FCI$, $\triangle ECA \cong \triangle LCB$ 等.

4.2.8

> 同命题 4.2.6 条件, 设 AE 交 BL 于点 M, 则四边形 $ECLM$ 是平行四边形.



证明

由 4.2.7 知:

AE 垂直平分 CF, 故 $\angle EAG = \dfrac{1}{2}\angle CAG = \angle CLG$, 即 $CL \mathbin{//} EM$.

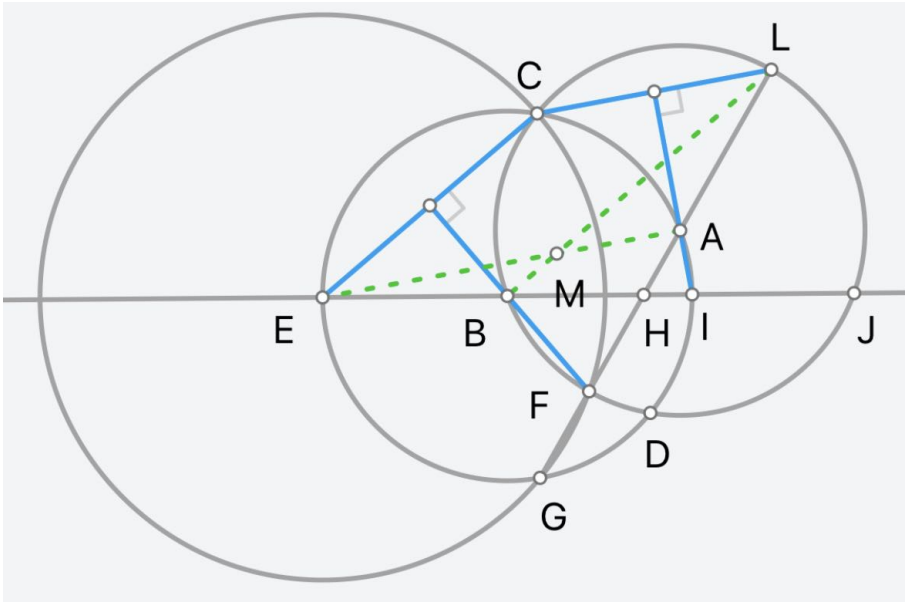
BL 垂直平分 CI, 故 $\angle CEI = \dfrac{1}{2}\angle CBI = \angle LBI$, 即 $EC \mathbin{//} ML$.

故四边形 $ECLM$ 是平行四边形. 证毕.

推论 设 EL 交 CM 于点 O, 则 $OE = OL, OC = OM$, 且 O 是 L, E, D 三点所在圆的圆心, 故 $OE = OL = OD$.

4.2.9

> 同命题 4.2.6 条件, FB 垂直平分 EC, IA 垂直平分 CL.



证明

由 4.2.8 知: 四边形 $ECLM$ 是平行四边形.

FL 是直径, 故 $FB \perp BL$, 而 $BL \ // \ EC$, 故 FB 垂直平分 EC.

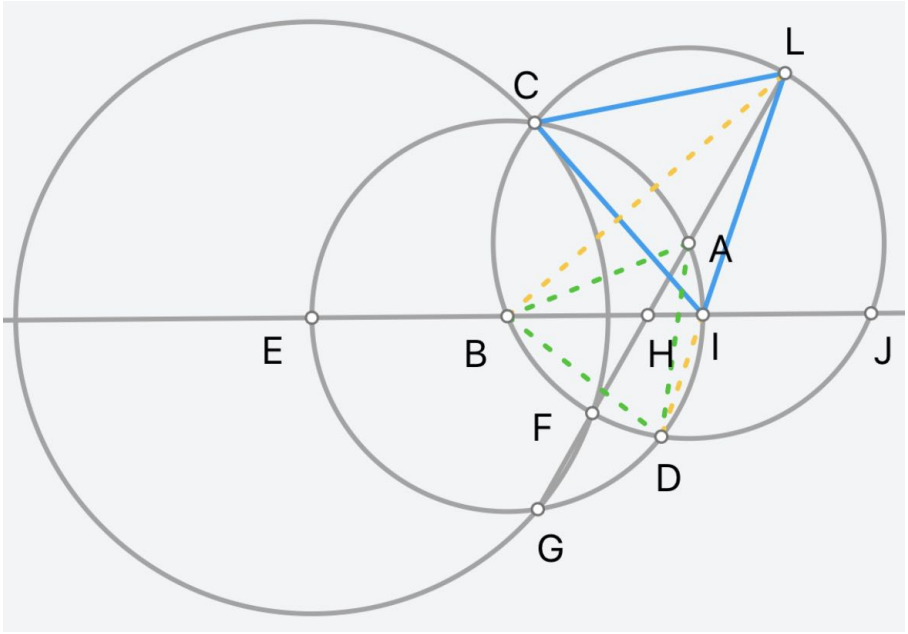
EI 是直径, 故 $IA \perp AE$, 而 $AE \ // \ CL$, 故 IA 垂直平分 CL.

证毕.

推论 $AI \ // \ CF$, 设 IA 交 CL 于点 A_1, 则四边形 $CMAA_1$ 是矩形. 同理 $IC \ // \ BF$ 并且也构成一个矩形.

4.2.10

> 同命题 4.2.6 条件, 则 D, I, L 三点共线.



证明

由 4.2.7 知: BL 垂直平分 CI.

$$\angle BLI = \frac{1}{2}\angle CLI = 30^\circ = \frac{1}{2}\angle BAD = \angle BLD,$$

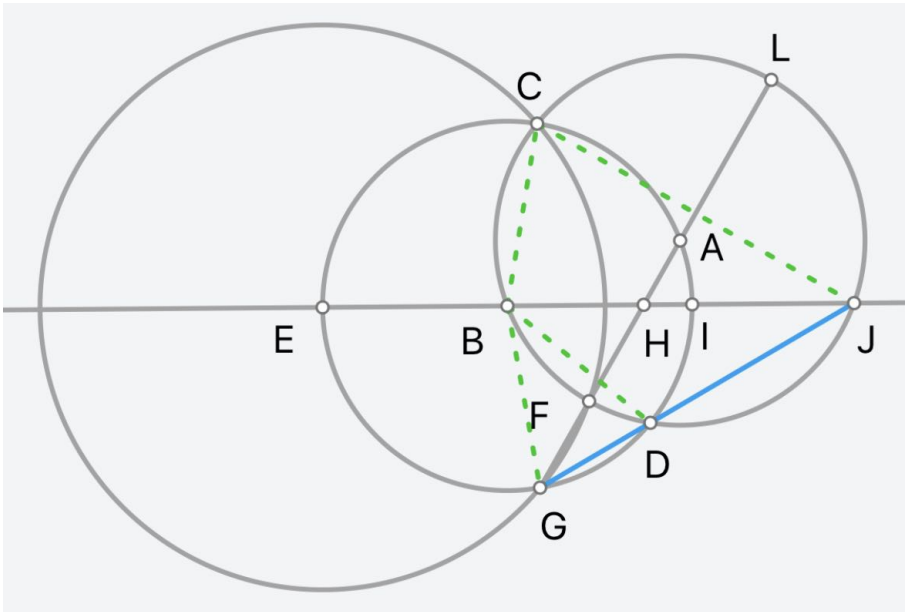
故 D, I, L 三点共线. 证毕.

推论 $DL \perp DE$.

注 连接 CH 可以得到对称的 60° 角、等比三角形和三点共线.

4.2.11

> 同命题 4.2.6 条件, 则 G, D, J 三点共线.



证明

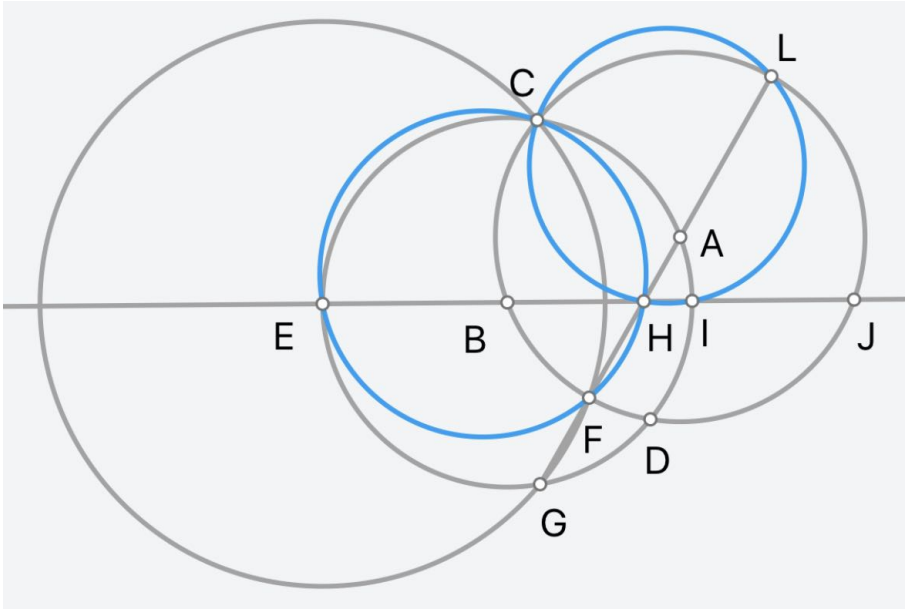
$$\angle BDG = \angle BGJ = \angle BCJ = 180^\circ - \angle BDJ,$$

故 G, D, J 三点共线. 证毕.

注 至此我们已得到四组非平凡三点共线: EFD, AFG, GIL, GDJ. 同理, 对称的还有四组非平凡三点共线.

4.2.12

> 同命题 4.2.6 条件, 则 E, F, H, C 四点共圆, H, C, L, I 四点共圆.



证明

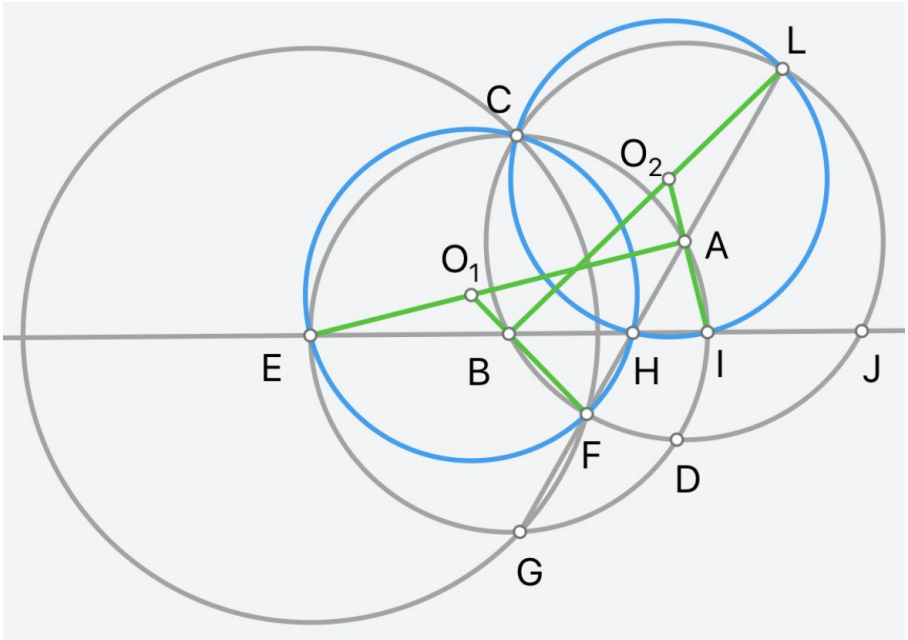
$\angle CHE = 60^\circ = \angle CFE$, 故 E, F, H, C 四点共圆;

$\angle CHL = 60^\circ = \angle CIL$, 故$H, C, L, I$ 四点共圆.

证毕.

4.2.13

> 同命题 4.2.6 条件, 设 E, F, H, C 所在圆的圆心为 O_1, H, C, L, I 所在圆的圆心为 O_2, 则有四组三点共线: AO_1E, BO_2L, O_1BF, O_2AI.



证明

由 4.2.7 知: AE 垂直平分 CF, BL 垂直平分 CI.

故 A, O_1, E 三点共线, B, O_2, L 三点共线.

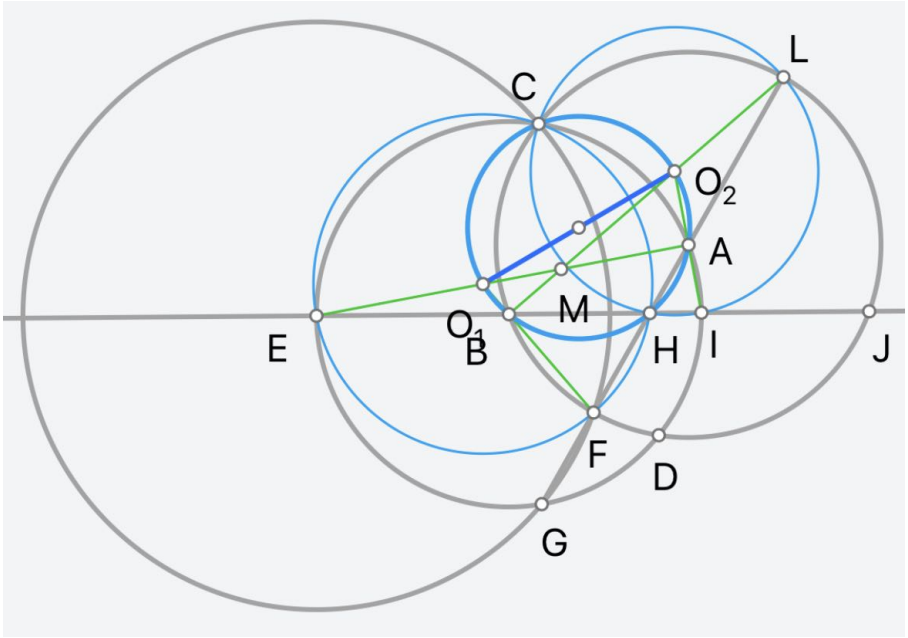
由 4.2.9 知: FB 垂直平分 EC, IA 垂直平分 CL.

故 O_1, B, F 三点共线, O_2, A, I 三点共线.

证毕.

4.2.14

> 同命题 4.2.13 条件, 则 A, B, O_1, O_2 四点共圆.



证明

由 4.2.13 知: O_1, B, F 三点共线, O_2, A, I 三点共线.

$\angle AO_1F = 180^\circ - \angle EO_1F = 60^\circ = \angle BHF$, 故 A, B, O_1, O_2 四点共圆.

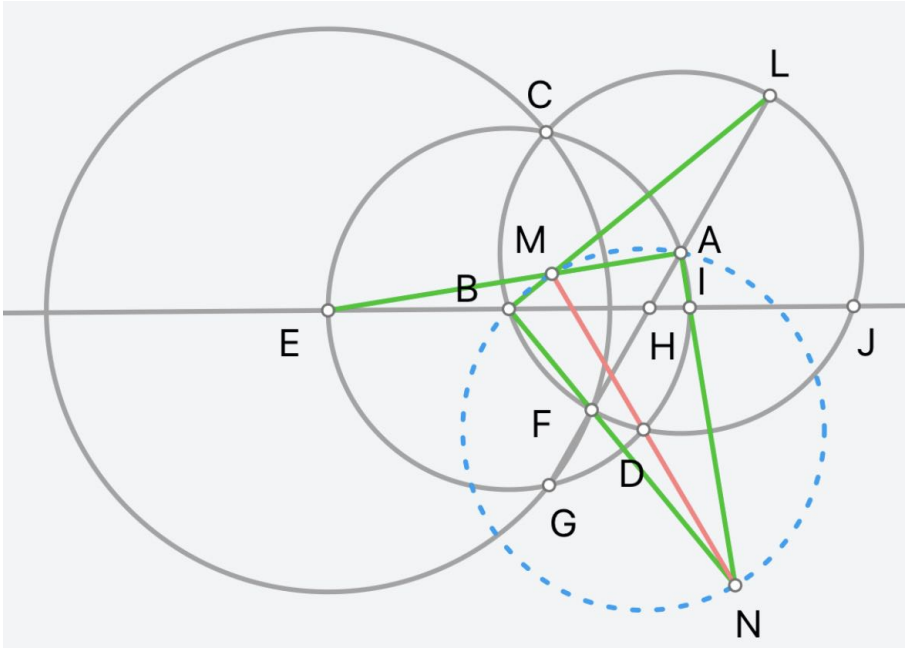
证毕.

推论

1. O_1, O_2, O_3 共线, 即 O_1O_2 是圆 O_3 的直径.
2. $\triangle FBH \backsim \triangle FAO_1$.
3. $\triangle BO_1M \backsim \triangle AO_2M$.
4. $\triangle IHA \backsim \triangle IO_2B$.

4.2.15

> 同命题 4.2.8 条件, 设 BF 交 AI 于点 N, 则 A, M, B, N 四点共圆, M, D, N 三点共线.



证明

$\angle MBN = \angle MAN = 90^\circ$, 故 A, M, B, N 四点共圆.

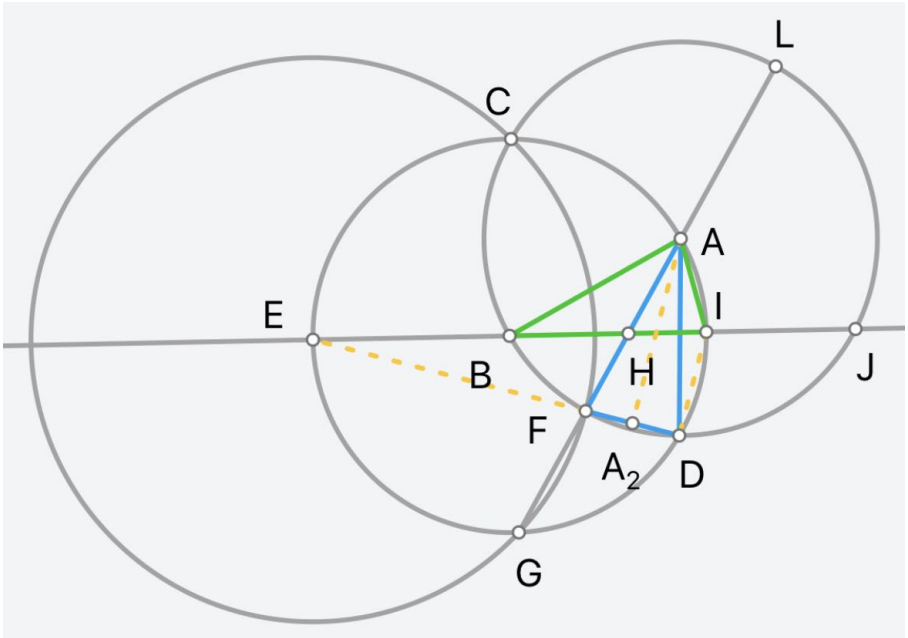
MN 为直径, 故 M, D, N 三点共线.

证毕.

推论 $DA = DB$, 故 D 为圆心, 且 $DM = DN = DA = DB = AB = BC = CA$.

4.2.16

> 同命题 4.2.3 条件, 则 $DF = AI$, $BF = ID$.



证明

作 $AA_2 \perp FD$ 交 FD 于点 A_2, 则 $\angle FAA_2 = \angle DAA_2$.

E, F, D 三点共线, IE 为直径, 故 $ID \perp FD$, $ID \parallel AA_2$.

故 $\angle ABI = 2\angle ADI = 2\angle DAA_2 = \angle FAD$,

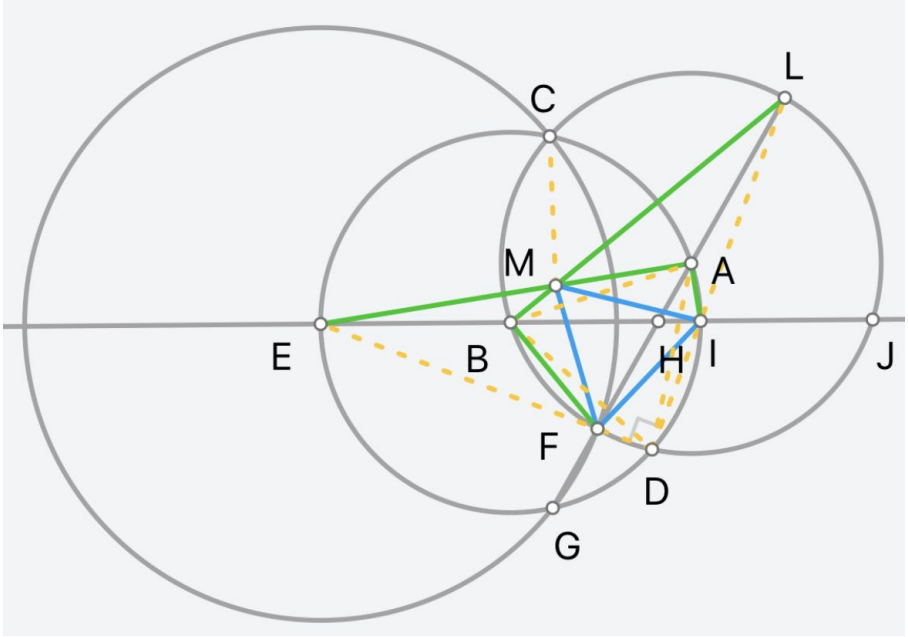
又 $AF=AD=BA=BI$, 故 $\triangle ABI\cong\triangle DAF$, $DF=AI$.

同理 $BF=ID$. 证毕.

推论 $\triangle AEI\cong\triangle DLF$, $AE=DL$. $\triangle BFD\cong\triangle DIA$.

4.2.17

> 同命题 4.2.8 条件, 则 $\triangle MIF$ 是等边三角形.



证明

FL, DE 和 EI 是直径, 故 $AE\perp AM$, $BL\perp BN$, $DE\perp DL$.

$\angle MBA=\angle MBF-\angle ABD-\angle DBF=30^\circ-\angle DBF=\angle FDB$,

$\angle MAB=180^\circ-\angle BMA-\angle MBA=30^\circ-\angle MBA=\angle FBD$,

又 $AB=BD$, 故 $\triangle AMB\cong\triangle BFD\cong\triangle DIA$.

故 $MB=FD=IA$, $BF=DI=AM$, 而 $\angle MBF=\angle FDI=\angle IAM=90^\circ$,

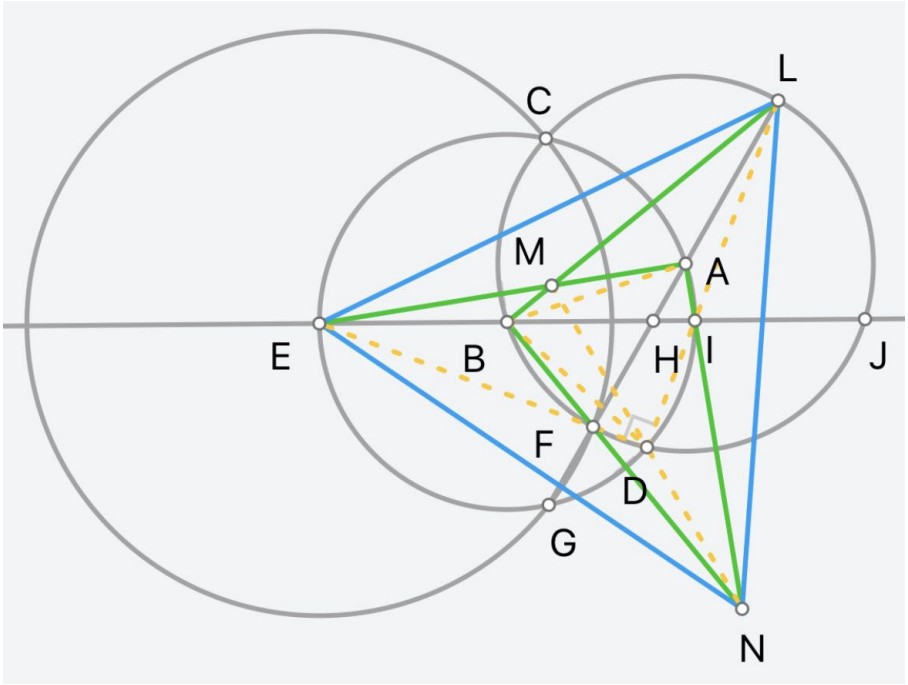
故 $\triangle MBF\cong\triangle FDI\cong\triangle IAM$, $MF=FI=IM$,

即 $\triangle MIF$ 是等边三角形. 证毕.

注 由 4.2.7 知: AE 垂直平分 CF, BL 垂直平分 CI, 故 $FM=MC=MI$. 由此可以知道 $\triangle MIF$ 是等腰三角形, 但是难以得知其它条件.

4.2.18

> 同命题 4.2.15 条件, $\triangle MLE$ 是等边三角形.



证明

由 4.2.17 知: $MB = FD = IA$,

又 $MN = FL = IE, \angle MBN = \angle FDL = \angle IAE = 90^\circ$,

故 $\triangle MBN \cong \triangle FDL \cong \triangle IAE, BN = DL = AE.$

由 4.2.17 知: $BF = DI = AM$,

又 $FL = IE = MN, \angle LBF = \angle EDI = \angle NAM = 90^\circ$,

故 $\triangle LBF \cong \triangle EDI \cong \triangle NAM, LB = ED = NA.$

又 $\angle LBN = \angle EDL = \angle NAE = 90^\circ$,

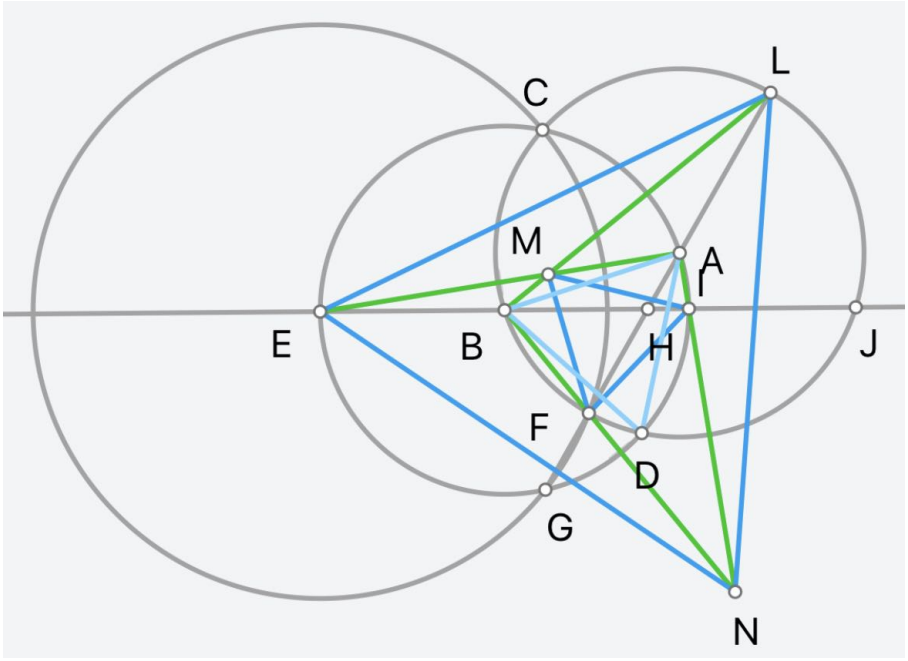
故 $\triangle LBN \cong \triangle EDL \cong \triangle NAE, LN = EL = NE,$

即 $\triangle MLE$ 是等边三角形, 证毕.

注 $\angle ANB = 30^\circ, \angle AMB = 150^\circ.$

4.2.19

> 同命题 4.2.15 条件, $\triangle MIF$, $\triangle MLE$ 和 $\triangle ABD$ 有着相同的几何中心.



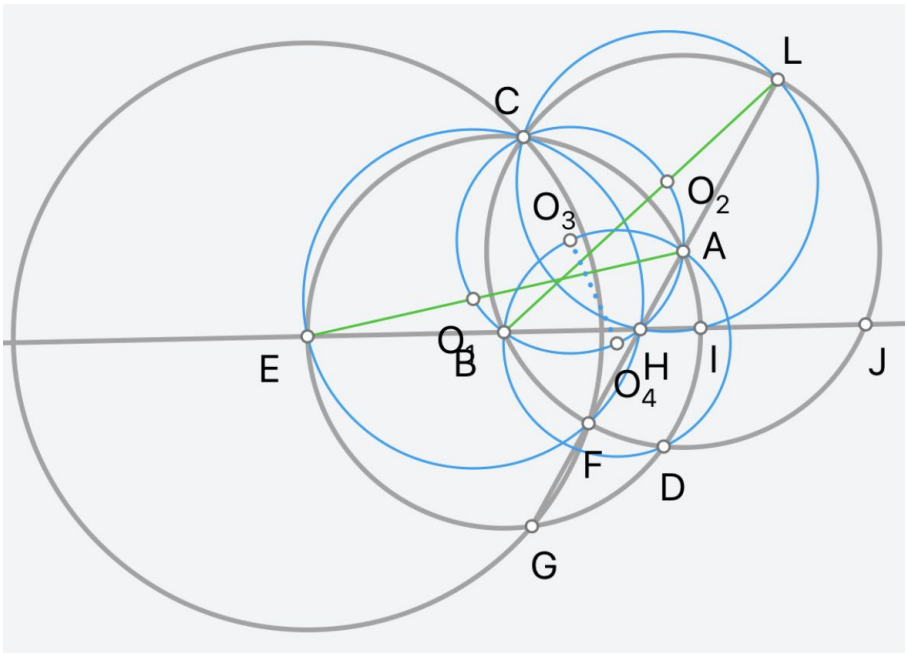
证明

由前两个命题的证明过程及对称性可知. (严谨的证明可以通过设其中一个三角形的外心, 并证明它也是另外两个三角形的外心.) 证毕.

注 这张图里有着许多全等三角形, 如 $\triangle AMF \cong \triangle BFI$, 不再一一列出.

4.2.20

> 同命题 4.2.13 和 4.2.15 条件, 则 $\triangle MIF$ 与 $\triangle MLE$ 的外心 O_4 落在圆 O_3 上.



证明

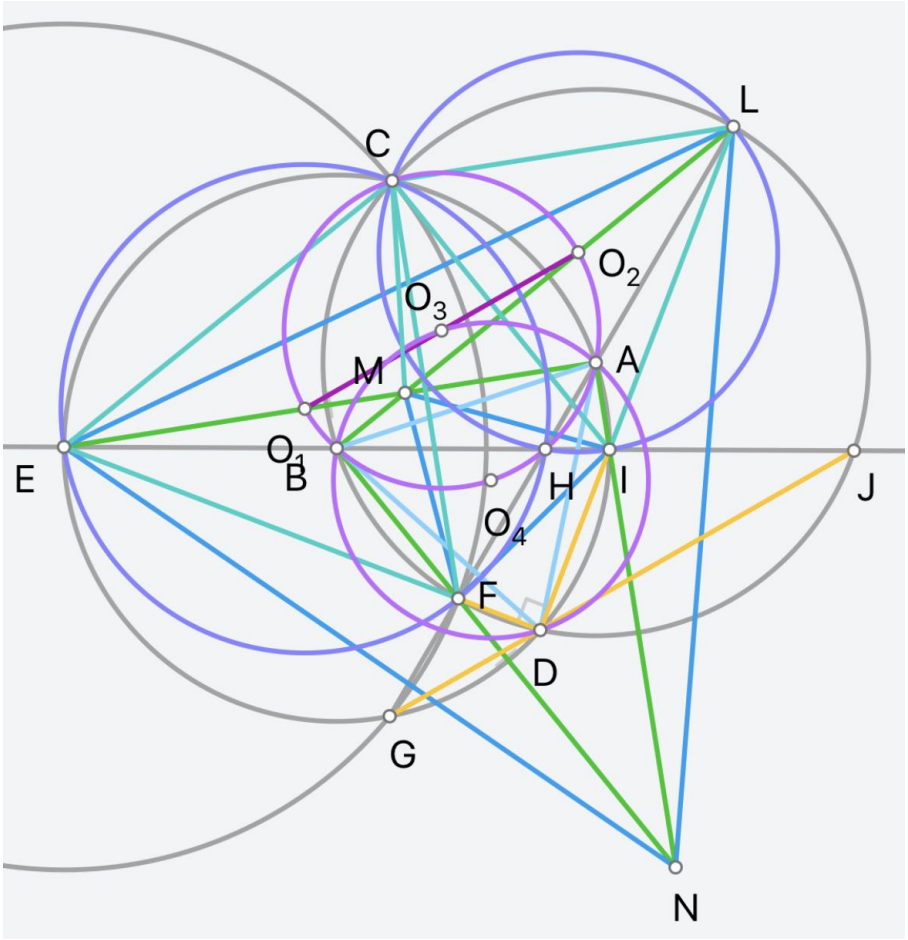
由 4.2.14 知: A, B, O_1, O_2 四点共圆, 故 $O_3A = O_3B = O_4A = O_4B$.

又 $\angle AO_4B = 2\angle ADB = 120^\circ$, 故 $\triangle O_3O_4A$ 和 $\triangle O_3O_4B$ 都是等边三角形,

$O_3O_4 = O_4B$, 故外心 O_4 落在圆 O_3 上.

推论 AB 与 O_3O_4 相互垂直平分.

附一张含有以上所有结论的图 (不包含辅助线)



A.2 引用

A.2.1 α

1. Let no one ignorant of geometry enter here.

   — Plato

2. Education is what remains after one has forgotten everything he learned in school.

   — Albert Einstein

3. Without geometry, life would be pointless.

   — Joke

4. You can't fit a round peg in a square hole.

   — American proverb

5. Without mathematics there is no art.

   — Luca Pacioli

6. Nature is an infinite sphere of which the center is everywhere and the circumference nowhere.

   — Blaise Pascal

7. Where there is matter, there is geometry.

— Johannes Kepler

A.2.2 β

1. The description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn.

   — Isaac Newton

2. Have the courage to use your own reason.

   — Immanuel Kant

3. Inspiration is needed in geometry, just as much as in poetry.

   — Alexander Pushkin

4. Contradiction is not a sign of falsity, nor the lack of contradiction a sign of truth.

   — Blaise Pascal

5. Measure twice, cut once!

   — Proverb

6. The knowledge of which geometry aims is the knowledge of the eternal.

   — Plato

7. What we know is not much. What we do not know is immense.

   — Pierre-Simon Laplace

8. If a man's wit be wandering, let him study the mathematics.

   — Francis Bacon

9. Intellectual labor in math lessons is the touchstone of thinking.

   — Vasili Sukhomlinsky

10. The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way.

    — Godfrey Hardy

A.2.3 γ

1. The study of the mathematics, like the Nile, begins in minuteness, but ends in magnificence.

   — Charles Colton

2. A mathematician is a machine for turning coffee into theorems.

   — Alfred Rényi

3. Knowledge exists to be imparted.

   — Ralph Emerson

4. All knowledge degenerates into probability.

   — David Hume

5. Knowledge comes, but wisdom lingers.

   — Alfred Tennyson

6. If there is a God, he is a great mathematician.

   — Paul Dirac

7. He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side.

   — Plato

8. To think deeply of simple things.

   — Arnold Ross

9. There is geometry in the humming of the strings.

   — Pythagoras

A.2.4 δ

1. A mathematical truth is neither simple nor complicated in itself, it is.

   — Emile Lemoine

2. The best angle from which to approach any problem is the try-angle.

   — Author Unknown

3. Logic is the anatomy of thought.

   — John Locke

4. We do not choose mathematics as our profession; it chooses us.

   — Yuri Manin

5. Mathematics is the most reliable form of prophecy.

   — Wilhelm Schwebel

6. The square root of 2 is also known as Pythagoras' constant.

   — Wikipedia

7. The square root of 3 is also known as Theodorus' constant.

   — Wikipedia

8. If we know anything we know it by mathematics.

   — Pierre Gassendi

9. For the first condition to be fulfilled in Mathematics is to be exact; the second is to be as clear and simple as possible.

   — Lazare Carnot

10. Thinking is the hardest work there is, which is probably the reason so few engage in it.

    — Henry Ford

11. The essential idea or method is always grasped intuitively long before any rational argument for the conclusion is devised.

    — Moriis Kline

A.2.5 ε

1. It is ignorance alone that could lead anyone to try to prove the axiom.

   — Aristotle

2. Logic is invincible, because in order to combat logic it is necessary to use logic.

   — Pierre Boutroux

3. Mathematics is a tool for reasoning.

   — Richard Feynman

4. Mathematics is the science of those things that are clear in and of themselves.

   — Carl Jacobi

5. No other science than mathematics strengthens so much our faith in the power of the human mind.

   — Hugo Steinhaus

6. Geometry is the real life.

   — Oscar Zariski

7. Most people would die sooner than think — in fact they do so.

   — Bertrand Russell

8. A round man cannot be expected to fit in a square hole right away. He must have time to modify his shape.

   — Mark Twain

9. If you wish to learn swimming you have to go into the water and if you wish to become a problem solver you have to solve problems.

   — George Pólya

10. As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

    — Joseph Louis Lagrange

A.2.6 ζ

1. Symmetry is one of the ideas by which man through the ages has tried to comprehend and create order, beauty, and perfection.

   — Hermann Weyl

2. Symmetry is what we see at a glance; based on the fact that there is no reason for any difference...

   — Blaise Pascal

3. Let no man who is not a Mathematician read the elements of my work.

   — Leonardo da Vinci

4. I would live to study, and not study to live.

   — Francis Bacon

5. There is nothing strange in the circle being the origin of any and every marvel.

   — Aristotle

6. Mathematics is a game played according to certain rules with meaningless marks on paper.

   — David Hilbert

7. One geometry cannot be more true than another; it can only be more convenient.

   — Henri Poincaré

8. The subject of mathematics is so serious that nobody should miss an opportunity to make it a little bit entertaining.

   — Blaise Pascal

9. Mathematics is the language with which God wrote the universe.

   — Galileo Galilei

10. Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not.

    — Godfrey Hardy

11. I would rather discover a single demonstration [in geometry] than become king of the Persians.

    — Democritus

A.2.7 η

1. The whole is greater than the sum of its parts.

   — Aristotle

2. We must know, we will know!

   — David Hilbert

3. The human mind has first to construct forms, independently, before we can find them in things.

   — Albert Einstein

4. Mathematics is gymnastics of thinking.

   — Alexander Suvorov

5. Life is good for only two things, discovering mathematics and teaching mathematics.

   — Simeon Poisson

6. I should almost therefore put forward the proposal that the third hypothesis (angle sum of a triangle less than two right angles) holds on the surface of an imaginary sphere.

   — Johann Lambert

7. Circles to square and cubes to double would give a man exercise trouble.

   — Matthew Prior

8. There are things which seem incredible to most men who have not studied Mathematics.

   — Archimedes

9. Do not worry about your difficulties in Mathematics: I can assure you mine are far greater.

   — Albert Einstein

10. There are two ways to do great mathematics. The first is to be smarter than everybody else. The second way is to be stupider than everybody else — but persistent.

    — Raoul Bott

11. It is easy to go from the house of reality to the forest of mathematics, but only few know how to go back.

    — Hugo Steinhaus

A.2.8 θ

1. He became a poet, he lacked imagination for a mathematician.

   — David Hilbert

2. If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy.

   — Alfred Rényi

3. I am ever more convinced that the necessity of our (Euclidean) geometry cannot be proved — at least not by human reason for human reason.

   — Carl Friedrich Gauss

4. The essence of mathematics is its freedom.

   — Georg Cantor

5. Good things, when short, are twice as good.

   — Baltasar Gracian

6. Could a dissertation of 2 lines deserve and get a Fellowship? In mathematics the answer is yes.

   — John Littlewood

7. Mathematics may, like poetry or music, 'promote and sustain a lofty habit of mind', and so increase the happiness of mathematicians and even of other people.

   — Godfrey Hardy

8. Math is not about following directions, it's about making new directions.

   — Paul Lockhart

9. Truth is ever to be found in simplicity, and not in the mutiplicity and confusion of things.

   — Isaac Newton

10. The art of doing mathematics consists of finding that special case which contains all the germs of generality.

    — David Hilbert

A.2.9 ι

1. Geometry is frozen music.

   — Johann Goethe

2. Everything around us is geometry.

   — Le Corbusier

3. Mathematics is less knowledge than skill.

   — Philip Davis

4. It is not enough to have a good mind. The main thing is to use it well.

   — Rene Descartes

5. Everything should be made as simple as possible, but not simpler.

   — Albert Einstein

6. It is hard to convince a high-school student that he will encounter a lot of problems more difficult than those of algebra and geometry.

   — Edgar Watson Howe

7. The roots of education are bitter, but the fruit is sweet.

   — Aristotle

8. Geometry is knowledge of the eternally existent.

   — Pythagoras

9. I think, therefore I am.

   — Rene Descartes

10. Neither thirty years, nor thirty centuries, affect the clearness, or the charm, of Geometrical truths.

    — Charles Dodgson

A.2.10 κ

1. Nothing is more important than to see the sources of invention which are, in my opinion more interesting than the inventions themselves.

   — Gottfried Leibniz

2. It is the glory of geometry that from so few principles, fetched from without, it is able to accomplish so much.

   — Isaac Newton

3. Imagination is more important than knowledge.

   — Albert Einstein

4. Imagination decides everything.

   — Blaise Pascal

5. Science is simply common sense at its best.

   — Thomas Huxley

6. Three Rules of Work: Out of clutter find simplicity; From discord find harmony; In the middle of difficulty lies opportunity.

   — Albert Einstein

7. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

   — Godfrey Hardy

8. Mathematics is the door and key to the sciences.

   — Roger Bacon

9. Knowledge is power.

   — Francis Bacon

10. Knowledge is light and ignorance is darkness.

    — Proverb

11. Education is the best provision for the journey to old age.

    — Aristotle

12. It is not once nor twice but times without number that the same ideas make their appearance in the world.

    — Aristotle

A.2.11 λ

1. Mathematics is concerned only with the enumeration and comparison of relations.

    — Friedrich Gauss

2. Mathematics is the part of physics where experiments are cheap.

    — Vladimir Arnold

3. The Golden ratio has inspired thinkers of all disciplines like no other number in the history of mathematics.

    — Mario Livio

4. The scientist is not a person who gives the right answers, he's one who asks the right questions.

    — Claude Levi-Strauss

5. It is impossible to be a mathematician without being a poet in soul.

    — Sofia Kovalevskaya

6. The mathematical sciences particularly exhibit order symmetry and limitations; and these are the greatest forms of the beautiful.

    — Aristotle

7. Mathematics is a language plus reasoning; it is like a language plus logic.

    — Richard Feynman

8. Mighty is geometry; joined with art, resistless.

    — Euripides

9. In any special doctrine of nature there can be only as much proper science as there is mathematics therein.

    — Immanuel Kant

10. In mathematics the art of proposing a question must be held of higher value than solving it.

    — Georg Cantor

A.2.12 μ

1. "Obvious" is the most dangerous word in mathematics.

    — Eric Bell

2. You are the only person on earth who can use your ability. It's an awesome responsibility.

    — Zig Ziglar

3. One quickly realizes that simplicity and geometry are the language of timelessness and universality.

    — Paul Rand

4. Music is the arithmetic of sounds as optics is the geometry of light.

    — Claude Debussy

5. There are no sects in geometry.

   — Voltaire

6. A mathematical proof should resemble a simple and clear-cut constellation, not a scattered cluster in the Milky Way.

   — Godfrey Hardy

7. Beauty is geometry.

   — Joanne Rowling

8. Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world.

   — Roger Bacon

9. The best way to make mathematics interesting to students and laymen is to approach it in a spirit of play.

   — Martin Gardner

10. A good mathematical joke is better, and better mathematics, than a dozen mediocre papers.

    — John Littlewood